

# Gauging the three-nucleon spectator equation

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## Abstract

We derive relativistic three-dimensional integral equations describing the interaction of the three-nucleon system with an external electromagnetic field. Our equations are unitary, gauge invariant, and they conserve charge. This has been achieved by applying the recently introduced gauging of equations method to the three-nucleon spectator equations where spectator nucleons are always on mass shell. As a result, the external photon is attached to all possible places in the strong interaction model, so that current and charge conservation are implemented in the theoretically correct fashion. Explicit expressions are given for the three-nucleon bound state electromagnetic current, as well as the transition currents for the scattering processes  $\gamma^3\text{He} \rightarrow \text{NNN}$ ,  $Nd \rightarrow \gamma Nd$ , and  $\gamma^3\text{He} \rightarrow Nd$ . As a result, a unified covariant three-dimensional description of the  $\text{NNN}-\gamma\text{NNN}$  system is achieved.

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## I. INTRODUCTION

The difficulty of solving four-dimensional scattering equations has led to a number of three-dimensional reduction schemes that preserve the covariance and unitarity of the original equations [1–5]. Here we shall be concerned with one of these schemes, that introduced by Gross [4], where some of the particles, typically the spectator particles of the given process, are restricted to be on their mass shell. The resultant three-dimensional equations are called the “spectator equations”. In the three-particle system, for example, the spectator particle is well defined (it’s the one flying past two interacting particles), and putting it on mass shell in every intermediate state results in the three-body spectator equations. The Gross approach has been used recently in successful relativistic calculations of nucleon-nucleon scattering [6], elastic electron-deuteron scattering [7], pion photoproduction from the nucleon [8], and the triton binding energy [9]. The quantities used or obtained from these calculations, such as the three-nucleon bound state wave function, one- and two-body interaction currents, etc., form just what would be needed to calculate the electromagnetic properties of the three-nucleon system. Unfortunately, the expressions needed to calculate such electromagnetic properties are not presently available.

The purpose of this paper is therefore to derive, within the framework of the spectator approach, gauge invariant expressions for the various electromagnetic transition currents of the three-nucleon system. In particular, we give expressions for the three-nucleon bound state current from which the triton or  ${}^3\text{He}$  electromagnetic form factors follow directly. We also derive expressions for the scattering processes  $\gamma^3\text{He} \rightarrow \text{NNN}$ ,  $Nd \rightarrow \gamma Nd$ , and  $\gamma^3\text{He} \rightarrow Nd$  (here, as in the rest of the paper, we use  ${}^3\text{He}$  as the generic symbol for a three-nucleon bound state).

The main tool of the derivation is the method of gauging equations introduced by us recently for four-dimensional equations [10], and for three-dimensional equations within the spectator approach [11]. This method results in electromagnetic amplitudes where the external photon is effectively coupled to every part of every strong interaction diagram in the model. Current and charge conservation are therefore implemented in the theoretically correct fashion. For the spectator approach, the gauging of equations method has two especially important features. Firstly, it avoids the difficulty of choosing the spectator particles in approaches where the photon is first coupled to hadrons at the level of four-dimensional quantum field theory. Once the hadronic spectator equations are specified, the gauging of equations method attaches photons in an automatic way, without the need for any new spectator particles to be introduced. Secondly, when applied to four-dimensional three-nucleon equations, the gauging of equations method has enabled us to avoid double counting of diagrams overlooked in previous works [10]. This means that in the present case of the spectator approach, such overcounting is likewise automatically avoided by the use of the gauging of equations method.

A key ingredient in our final expressions is  $\delta^\mu$ , the gauged on-mass-shell propagator for the nucleon. Knowledge of an explicit form for  $\delta^\mu$  that satisfies both the Ward-Takahashi identity and the Ward identity is essential for the gauge invariance and charge conservation properties of the three-nucleon electromagnetic currents presented in this paper. Such a  $\delta^\mu$  that satisfies both these identities has been presented in Refs. [11] and [12]. Thus we have brought together all the expressions necessary for a covariant, unitary, gauge invariant, and charge conserving three-dimensional calculation of the electromagnetic properties of the three-nucleon system.

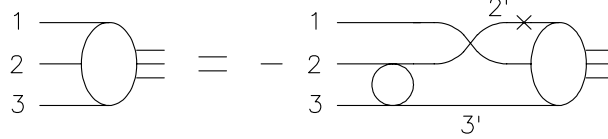


FIG. 1. Illustration of Eq. (3) for the bound state vertex function  $\Phi_1$ . The on mass shell particle is indicated by a cross.

## II. GAUGING THE THREE-NUCLEON BOUND STATE EQUATION

### A. The spectator equation

In this presentation we work within the framework of the spectator equations for three identical particles in the absence of three-body forces [4]. In this formalism two of the three particles are restricted to their mass shell by the following replacement of the usual Feynman propagator  $d(p)$

$$d(p) = \frac{i\Lambda(p)}{p^2 - m^2 + i\epsilon} \quad \rightarrow \quad \delta(p) = 2\pi\Lambda(p)\delta^+(p^2 - m^2) \quad (1)$$

where  $\Lambda(p) = 1$  or  $\not{p} + m$  for scalar and spinor particles respectively, and  $\delta^+(p^2 - m^2)$  is the positive energy on-mass-shell  $\delta$ -function. We refer to  $\delta(p)$  as the “on-mass-shell particle propagator”.

In the four-dimensional formalism of quantum field theory, we may write the three-body bound state equation in symbolic form as [9]

$$\Phi_1 = -t_1 D P_{12} \Phi_1 \quad (2)$$

where  $\Phi_1$  is the Faddeev component of the bound-state vertex function (from now on simply called “the bound-state vertex function”) describing the contribution to the bound state from all processes where the (23) pair interacts last,  $t_1$  is the off-shell scattering amplitude of the (23) pair,  $D = d_2 d_3$  is the propagator of the (23) pair, and  $P_{12}$  is the operator interchanging particles 1 and 2. Note that our  $t_1$  is fully antisymmetric while  $\Phi_1$  is antisymmetric only under the interchange of its 2nd and 3rd particle labels.<sup>1</sup> Because of these symmetries, one can equally well use  $P_{13}$  in Eq. (2) instead of  $P_{12}$  without changing the value of  $\Phi_1$ . Once the  $P_{12}$  form is chosen as in Eq. (2), the bound state “spectator equation” is obtained from Eq. (2) by putting particle 2 on the mass shell in intermediate state, i.e. by the replacement  $d_2 \rightarrow \delta_2$  in  $D$ :

$$\Phi_1 = -t_1 \delta_2 d_3 P_{12} \Phi_1. \quad (3)$$

The explicit numerical form of Eq. (3) is given in the Appendix, see Eq. (A1). Had we chosen the  $P_{13}$  form of the bound state equation, the spectator equation would instead be defined by putting particle 3 on mass shell; however, the solution obtained would be identical to that

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<sup>1</sup>Eq. (2) differs by a factor  $-2$  from the corresponding equation in Ref. [9] due to the use of different conventions for  $t$  and  $d$ .

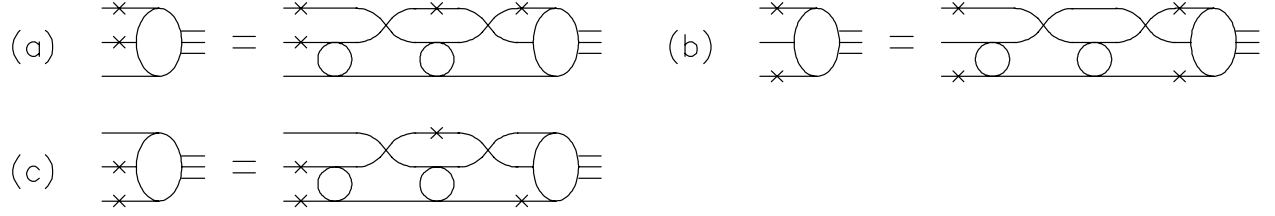


FIG. 2. All the possible three-nucleon bound state equations with two particles on mass shell. (a) The spectator equation as given in Eq. (5). (b) The bound state equation given in Eq. (6). (c) The bound state equation given in Eq. (7).

obtained from Eq. (3). We illustrate Eq. (3) in Fig. 1. Of course to get a closed three-dimensional equation for  $\Phi_1$  it is necessary to also put the external particles 1 and 2 on mass shell in Eq. (3).

It is useful to point out that the spectator equation is not the only possible three-dimensional equation that follows from Eq. (2) by putting two particles on mass shell (in three-body intermediate states). However, it is the best one. Indeed we can investigate all the possibilities by iterating Eq. (2) once:

$$\begin{aligned}\Phi_1 &= t_1 d_2 d_3 P_{12} t_1 d_2 d_3 P_{12} \Phi_1 \\ &= t_1 d_3 P_{12} t_1 P_{12} d_1 d_2 d_3 \Phi_1\end{aligned}\quad (4)$$

thereby obtaining an equation for  $\Phi_1$  with the compact kernel  $t_1 d_3 P_{12} t_1 P_{12} = t_1 d_3 t_2$ . Eq. (4) shows that there are only three possibilities to restrict two of the three intermediate state particles to their mass shells:

$$(a) \quad \Phi_1 = t_1 d_3 t_2 \delta_1 \delta_2 d_3 \Phi_1, \quad (5)$$

$$(b) \quad \Phi_1 = t_1 d_3 t_2 \delta_1 d_2 \delta_3 \Phi_1, \quad (6)$$

$$(c) \quad \Phi_1 = t_1 d_3 t_2 d_1 \delta_2 \delta_3 \Phi_1. \quad (7)$$

Eq. (5) is just the first iteration of the spectator equation Eq. (3). After setting the external particles 1 and 2 on mass shell, the two-body t-matrices in Eq. (5) have two legs on shell and two legs off shell, and therefore depend on one parameter, the off-mass-shell energy, just like two-body t-matrices in quantum mechanics. Eqs. (6) and (7), on the other hand, are not iterations of any form similar to Eq. (3) with a kernel linear in  $t_1$ . Moreover, after setting two of the external particles on mass shell to get closed equations, the kernels  $t_1 d_3 t_2$  in Eqs. (6) and (7) suffer a major drawback in that one of the t-matrices  $t_1$  or  $t_2$  has three legs that are off mass shell. These observation can be seen explicitly in the illustrations of Eqs. (5)-(7) given in Fig. 2.

## B. Gauging the spectator equation

The question of how to couple an external electromagnetic field to a system of hadrons described by four-dimensional integral equations, and still retain gauge invariance, has now been solved. On the two-particle level the problem was first solved by Gross and Riska [13]

who showed that the one-body current combined with the gauged interaction kernel of the two-body Bethe-Salpeter equation gives a gauge invariant two-body current. Similar progress was made by van Antwerpen and Afnan [14] who showed how to construct a gauge invariant current for the relativistic  $\pi N$  system where pion absorption can take place. More recently, we have introduced a general method where any system described by integral equation can be gauged [10]. The method involves the idea of gauging the integral equations themselves, and results in an electromagnetic current where the photon is coupled to all possible places in all possible strong interaction Feynman graphs of the model. We have applied the gauging of equations method to the relativistic three-nucleon system, thereby solving an overcounting problem that had previously been overlooked [10]. In this section we would like to apply our method to gauge the bound state spectator equation, Eq. (3), in order to obtain a relativistic gauge invariant three-dimensional description of the three-nucleon bound state current.

In our procedure, we do not use the on-mass-shell  $\delta$ -function to eliminate the zero'th component of the spectator internal momentum in Eq. (3) until after the gauging of the equation is done. Instead we follow the method outlined in Ref. [11] and treat Eq. (3) as an eight-dimensional Bethe-Salpeter equation where some of the propagators are represented by on-mass-shell  $\delta$ -functions. This enables us to apply our method of gauging in just the same way as was done for the eight-dimensional case of Eq. (2) [10]. Gauging Eq. (3) in this way, it immediately follows that

$$\Phi_1^\mu = -t_1 \delta_2 d_3 P_{12} \Phi_1^\mu - (t_1^\mu \delta_2 d_3 + t_1 \delta_2^\mu d_3 + t_1 \delta_2 d_3^\mu) P_{12} \Phi_1. \quad (8)$$

It is clear from the form of this equation that the quantity  $\Phi_1^\mu$  corresponds to that part of the  ${}^3\text{He} \rightarrow NNN$  electromagnetic transition current where the (23) pair was last to interact, and where no photons are attached to the external constituent legs (a rigorous proof of this statement was given for the case of four-dimensional quantum field theory in Ref. [10]). In this respect we note that the bound state vertex component  $\Phi_1$  is a purely nonperturbative object and as such cannot be represented as a sum of diagrams; nevertheless,  $\Phi_1^\mu$  can be formally considered as  $\Phi_1$  with photons attached everywhere “inside”. Note that Eq. (8) is an integral equation for  $\Phi_1^\mu$  with  $\Phi_1$  being an input. Another input is the gauged Feynman propagator  $d_3^\mu$ . For particle  $i = 1, 2$ , or  $3$ , the gauged Feynman propagator  $d_i^\mu$  is defined by

$$d_i^\mu(p', p) = d_i(p') \Gamma_i^\mu(p', p) d_i(p) \quad (9)$$

where  $\Gamma_i^\mu(p', p)$  is the particle's electromagnetic vertex function. For a structureless nucleon of charge  $e_i$ ,  $\Gamma_i^\mu(p', p) = e_i \gamma_i^\mu$ . A further input in Eq. (8) is  $\delta_2^\mu$ , the gauged on-mass-shell propagator of particle 2. As shown in Ref. [11], taking the explicit form

$$\delta^\mu(p', p) = 2\pi i \Lambda(p') \Gamma^\mu(p', p) \Lambda(p) \frac{\delta^+(p'^2 - m^2) - \delta^+(p^2 - m^2)}{p^2 - p'^2} \quad (10)$$

for the gauged on-mass-shell propagator  $\delta^\mu$ , ensures current and charge conservation of our final results. This is a consequence of the fact that the  $\delta^\mu$  of Eq. (10) satisfies both the Ward-Takahashi identity

$$(p'_\mu - p_\mu) \delta^\mu(p', p) = ie [\delta(p) - \delta(p')] \quad (11)$$

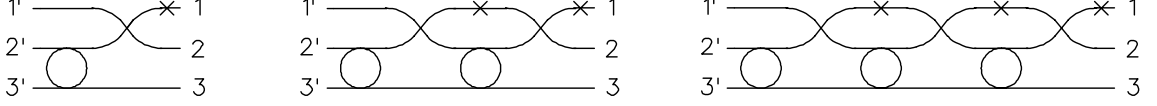


FIG. 3. Illustration of the first three iterations of the equations for  $X$ , Eqs. (15). That it is the spectator particle that is on mass shell is clearly visible.

as well as the Ward identity

$$\delta^\mu(p, p) = -ie \frac{\partial \delta(p)}{\partial p_\mu}, \quad (12)$$

and that Eq. (8) gives an expression for  $\Phi_1^\mu$  which has photons coupled everywhere.

We may formally solve Eq. (8) to obtain

$$\Phi_1^\mu = -(1 + t_1 \delta_2 d_3 P_{12})^{-1} (t_1^\mu \delta_2 d_3 + t_1 \delta_2^\mu d_3 + t_1 \delta_2 d_3^\mu) P_{12} \Phi_1. \quad (13)$$

The factor  $(1 + t_1 \delta_2 d_3 P_{12})^{-1}$  in this equation clearly describes the final state  $NNN \rightarrow NNN$  process. Defining

$$X = (1 + t_1 \delta_2 d_3 P_{12})^{-1} \quad (14)$$

it follows that  $X$  satisfies the two equations

$$X = 1 - t_1 \delta_2 d_3 P_{12} X; \quad X = 1 - X t_1 \delta_2 d_3 P_{12}. \quad (15)$$

As expected, these are three-nucleon scattering equations whose kernel is identical to that of the bound state equation, Eq. (3). We illustrate the first three iterations of these equations in Fig. 3. It is evident that  $X$  consists of all possible  $NNN \rightarrow NNN$  diagrams where the (13) nucleon pair is first to interact and the (23) pair is last to interact.

### C. Three-body bound state current

We recall that  $\Phi_1^\mu$  describes the  ${}^3\text{He} \rightarrow NNN$  electromagnetic transition current where the (23) nucleon pair is last to interact, and where no photons are attached to the final state nucleon legs. As such, it contains all the information that is necessary to specify the three-nucleon bound state interaction current  $j^\mu$ . Indeed, we shall use the expression for  $\Phi_1^\mu$  given in Eq. (13) to extract  $j^\mu$ . The key observation about Eq. (13) is that the final state interaction term  $X = (1 + t_1 \delta_2 d_3 P_{12})^{-1}$  has a pole at  $K^2 = M^2$  where  $K$  is the total four-momentum and  $M$  is the mass of the three-nucleon bound state. This follows from the fact that the equations for  $X$  and  $\Phi_1$  have the same kernel and that the solution for  $\Phi_1$  exists. The three-body bound state current then follows by taking the residue of Eq. (13) at this pole.

We write the pole structure of  $X$  as

$$X(K; p_1 p_2, q_1 q_2) \sim i \frac{\Phi_1^K(p_1 p_2) \bar{\Psi}_2^K(q_1 q_2)}{K^2 - M^2} \quad \text{as} \quad K^2 \rightarrow M^2, \quad (16)$$

which defines the quantity  $\bar{\Psi}_2$ . In order to determine  $\bar{\Psi}_2$ , we take residues of Eqs. (15) at the three-nucleon bound state pole, thereby obtaining the equations

$$\Phi_1 = -t_1 \delta_2 d_3 P_{12} \Phi_1; \quad \bar{\Psi}_2 = -\bar{\Psi}_2 t_1 \delta_2 d_3 P_{12}. \quad (17)$$

The first of these is the bound state equation for  $\Phi_1$ , which of course is the reason that Eq.(16) was written with a  $\Phi_1$  factor. The second equation can be written as

$$\bar{\Psi}_2 = -\bar{\Psi}_2 P_{12} t_2 \delta_1 d_3 \quad (18)$$

which has the same form as the equation for the second Faddeev component of the bound state wave function in four-dimensional quantum field theory [10], hence our choice of notation for  $\bar{\Psi}_2$ . However, in contrast to the four-dimensional quantum field theory case, the  $\bar{\Psi}_2$  of Eq. (18) contains explicit on-mass shell propagators. This can already be seen from Eq. (18) where the  $\delta_1$  that is present on the r.h.s. contains a  $\delta$ -function that is not integrated over. But the full structure of  $\bar{\Psi}_2$  becomes clear only after we iterate Eq. (18) once, obtaining

$$\bar{\Psi}_2 = \bar{\Psi}_2 P_{12} t_2 \delta_1 d_3 P_{12} t_2 \delta_1 d_3 = \bar{\Psi}_2 P_{12} t_2 d_3 t_1 P_{12} \delta_1 \delta_2 d_3. \quad (19)$$

This reveals an explicit factor  $\delta_1 \delta_2 d_3$  with two on-mass-shell propagators, followed by the connected term  $t_2 d_3 t_1$ . Thus  $\bar{\Psi}_2$  has a structure of the form

$$\bar{\Psi}_2 = -\bar{\Phi}_1 P_{12} \delta_1 \delta_2 d_3 \quad (20)$$

where

$$\bar{\Phi}_1 = -\bar{\Psi}_2 P_{12} t_2 d_3 t_1 \quad (21)$$

has no propagators on its three external legs. Multiplying Eq. (18) on the right by  $-P_{12} t_2 d_3 t_1$ , we find that  $\bar{\Phi}_1$  satisfies the equation

$$\bar{\Phi}_1 = -\bar{\Phi}_1 P_{12} \delta_2 d_3 t_1. \quad (22)$$

This is the conjugate equation to Eq. (3), hence our choice of notation for  $\bar{\Phi}_1$  in Eq. (20). With  $\bar{\Phi}_1$  and  $\bar{\Psi}_2$  determined by Eqs. (20) and (22), the residue of Eq. (16) is completely specified.

Thus, in the vicinity of the three-body bound state pole ( $K^2 \rightarrow M^2$ ),  $\Phi_1^\mu$  behaves as

$$\Phi_1^\mu \sim -i \frac{\Phi_1 \bar{\Phi}_1 P_{12} \delta_1 \delta_2 d_3}{K^2 - M^2} (t_1^\mu \delta_2 d_3 + t_1 \delta_2^\mu d_3 + t_1 \delta_2 d_3^\mu) P_{12} \Phi_1. \quad (23)$$

The three-nucleon bound state current in quantum field theory is given by the matrix element  $\langle K | J^\mu(0) | Q \rangle$  of the electromagnetic current operator  $J^\mu$  between momentum eigenstates  $|K\rangle$  and  $|Q\rangle$ . In the spectator approximation it can be determined by taking the residue of Eq. (23) at the three-nucleon bound state pole on the left:

$$\langle K | J^\mu(0) | Q \rangle \equiv j^\mu(K, Q) = \bar{\Phi}_1^K P_{12} \delta_1 \delta_2 d_3 (t_1^\mu \delta_2 d_3 + t_1 \delta_2^\mu d_3 + t_1 \delta_2 d_3^\mu) P_{12} \Phi_1^Q. \quad (24)$$

Here  $K$  and  $Q$  are the total four-momenta of the final and initial bound states, respectively, with  $K^2 = Q^2 = M^2$  and  $K = Q + q$  where  $q$  is the four-momentum of the incoming photon. One can eliminate  $t_1$  from this expression by using Eq. (22), in this way obtaining

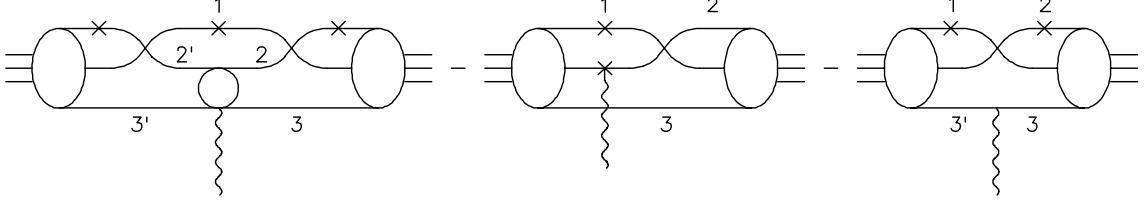


FIG. 4. The three-nucleon bound state current in the spectator model as given by Eq. (25). Particle labels correspond to those used in Eq. (32).

$$j^\mu(K, Q) = \bar{\Phi}_1^K P_{12} \delta_1 \delta_2 d_3 t_1^\mu \delta_2 d_3 P_{12} \Phi_1^Q - \bar{\Phi}_1^K \delta_1 (\delta_2^\mu d_3 + \delta_2 d_3^\mu) P_{12} \Phi_1^Q. \quad (25)$$

This expression is illustrated in Fig. 4. Note that the last two terms do not give the full one-body contribution to the bound state current as a further contribution comes from the gauged propagators inside  $t_1^\mu$ .

To find  $t_1^\mu$ , we first need to specify the spectator equations for  $t_1$ :

$$t_1 = v_1 + \frac{1}{2} v_1 \delta_2 d_3 t_1; \quad t_1 = v_1 + \frac{1}{2} t_1 \delta_2 d_3 v_1. \quad (26)$$

By gauging these equations one can express  $t_1^\mu$  in terms of the interaction current  $v_1^\mu$  as

$$t_1^\mu = \frac{1}{2} t_1 (\delta_2^\mu d_3 + \delta_2 d_3^\mu) t_1 + \left(1 + \frac{1}{2} t_1 \delta_2 d_3\right) v_1^\mu \left(1 + \frac{1}{2} \delta_2 d_3 t_1\right). \quad (27)$$

Note that our  $v_1$  is the sum of all possible irreducible diagrams for the scattering of two identical particles, therefore  $P_{23} v_1 = v_1 P_{23} = -v_1$ . That is why we do not need to use the symmetrised propagator  $\frac{1}{2}(\delta_2 d_3 + d_2 \delta_3)$  in Eq. (26) in order to satisfy the Pauli exclusion principle.

Although Eq. (25) may be the most practical equation for numerical calculations, with the help of Eq. (27) we can also eliminate  $t_1^\mu$  in favour of the interaction current  $v_1^\mu$ :

$$j^\mu(K, Q) = \bar{\Phi}_1^K \delta_1 (\delta_2^\mu d_3 + \delta_2 d_3^\mu) \left(\frac{1}{2} - P_{12}\right) \Phi_1^Q + \bar{\Phi}_1^K \left(P_{12} - \frac{1}{2}\right) \delta_2 d_3 \delta_1 v_1^\mu \delta_2 d_3 (P_{12} - \frac{1}{2}) \Phi_1^Q. \quad (28)$$

It is interesting to compare Eq. (28) with the corresponding expression obtained by using the same gauging method in the case of four-dimensional quantum field theory [10]:

$$j^\mu(K, Q) = \bar{\Phi}_1^K d_1 (d_2^\mu d_3 + d_2 d_3^\mu) \left(\frac{1}{2} - P_{12}\right) \Phi_1^Q + \bar{\Phi}_1^K \left(P_{12} - \frac{1}{2}\right) d_2 d_3 d_1 v_1^\mu d_2 d_3 (P_{12} - \frac{1}{2}) \Phi_1^Q. \quad (29)$$

This comparison makes clear the prescription  $d_1 \rightarrow \delta_1$ ,  $d_2 \rightarrow \delta_2$ ,  $d_1^\mu \rightarrow \delta_1^\mu$ ,  $d_2^\mu \rightarrow \delta_2^\mu$  that one should use to obtain the three-body bound state electromagnetic current in the three-dimensional spectator approach, Eq. (28), from the corresponding four-dimensional expression of Eq. (29).

In the impulse approximation where the interaction current  $v_1^\mu$  is neglected, we have that

$$j^\mu(K, Q) = \bar{\Phi}_1^K \delta_1 (\delta_2^\mu d_3 + \delta_2 d_3^\mu) \left(\frac{1}{2} - P_{12}\right) \Phi_1^Q. \quad (30)$$

This of course is the full one-body contribution to the bound state current. Because of propagator  $\delta_1$  in this expression, particle 1 is on mass shell (of course to the right of operator  $P_{12}$  this



on-mass shell particle becomes particle 2). The first term on the r.h.s of Eq. (30) also contains the gauged propagator  $\delta_2^\mu$ , and therefore according to Eq. (10), particle 2 can be off mass shell either to the left or to the right of the photon. Thus to calculate this first term, one needs to know  $\bar{\Phi}_1^K$  and  $\Phi_1^Q$  where only one external particle is on mass shell. These can always be determined from the spectator bound state vertex functions where two particles are on mass shell by using Eq. (3) and Eq. (22). Choosing the momenta of particles 1 and 2 as independent variables, we may write Eq. (30) in the explicit numerical form

$$j^\mu(K, Q) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^K(\bar{p}_1, p_2 + q) \delta(p_1) \delta^\mu(p_2 + q, p_2) d(Q - p_1 - p_2) \left(\frac{1}{2} - P_{12}\right) \Phi_1^Q(\bar{p}_1, p_2) \\ + \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^K(\bar{p}_1, \bar{p}_2) \delta(p_1) \delta(p_2) d^\mu(K - p_1 - p_2, Q - p_1 - p_2) \left(\frac{1}{2} - P_{12}\right) \Phi_1^Q(\bar{p}_1, \bar{p}_2) \quad (31)$$

where the momenta which are on-mass-shell are labelled with a bar over the top.

#### D. Gauge invariance

As the gauging of equations method effectively couples photons everywhere in the strong interaction model, gauge invariance is guaranteed. Nevertheless, here we would like to check this explicitly on our derived expression for the bound state current of Eq. (25).

Writing this equation out in full numerical form we have that

$$j^\mu(K, Q) = \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \frac{d^4 p'_2}{(2\pi)^4} \bar{\Phi}_1^K(p'_2 p_1 p'_3) \delta(p_1) \delta(p'_2) d(p'_3) t^\mu(p'_2 p'_3, p_2 p_3) \delta(p_2) d(p_3) \Phi_1^Q(p_2 p_1 p_3) \\ - \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^K(p_1 p'_2 p_3) \delta(p_1) \delta^\mu(p'_2, p_2) d(p_3) \Phi_1^Q(p_2 p_1 p_3) \\ - \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^K(p_1 p_2 p'_3) \delta(p_1) \delta(p_2) d^\mu(p'_3, p_3) \Phi_1^Q(p_2 p_1 p_3), \quad (32)$$

where  $p_3 = Q - p_1 - p_2$ ,  $p'_3 = K - p_1 - p'_2$ , and it is understood that  $p'_2 + p'_3 = p_2 + p_3 + q$  in the first integral,  $p'_2 = p_2 + q$  in the second integral, and  $p'_3 = p_3 + q$  in the third. Here we have followed the notation of Ref. [10] and displayed the momentum of each particle explicitly. Each of the gauged inputs in Eq. (32) satisfies a Ward-Takahashi identity (WTI). In the notation of Eq. (32), the WTI for  $t^\mu$  takes the form

$$q_\mu t^\mu(p'_2 p'_3, p_2 p_3) = i [e_2 t(p'_2 - q, p'_3; p_2 p_3) - t(p'_2 p'_3; p_2 + q, p_3) e_2] \\ + i [e_3 t(p'_2, p'_3 - q; p_2 p_3) - t(p'_2 p'_3; p_2, p_3 + q) e_3], \quad (33)$$

while for  $\delta^\mu$  and  $d^\mu$  the WTI's are

$$q_\mu \delta^\mu(p'_2, p_2) = i e_2 [\delta(p_2) - \delta(p'_2)], \quad (34)$$

$$q_\mu d^\mu(p'_3, p_3) = i e_3 [d(p_3) - d(p'_3)]. \quad (35)$$

In the present case of three nucleons, the charges  $e_i$  ( $i = 1, 2, 3$ ) are given by  $e_i = \frac{1}{2}[1 + \tau_3^{(i)}]e_p$  where  $\tau_3$  is the Pauli matrix for the third component of isospin, and  $e_p$  is the charge of the proton.

In order to prove gauge invariance of the bound state current, we follow the same procedure as we used for the distinguishable particle case [10], and evaluate the quantity  $q_\mu j^\mu$  by using the above WTI's in Eq. (32). However, unlike in the distinguishable particle case, subtle use of identical particle symmetry also needs to be made before the final expression is reduced to zero. Although this is straightforward, working with lengthy numerical expression like that of Eq. (32) tends to obscure the presentation. For this reason, here we would prefer to avoid the use of explicit numerical forms and instead to keep all our equations at the symbolic level. In order to write the WTI's of Eqs. (33)-(35) in symbolic form, we introduce the quantities  $\hat{e}_i$  whose numerical form is defined by

$$\hat{e}_i(p'_1 p'_2 p'_3, p_1 p_2 p_3) = ie_i (2\pi)^{12} \delta^4(p'_i - p_i - q) \delta^4(p'_j - p_j) \delta^4(p'_k - p_k), \quad (36)$$

where  $ijk$  are cyclic permutations of 123. Then the above WTI's can be written symbolically in terms of commutators as

$$q_\mu t_1^\mu = [\hat{e}_2, t_1] + [\hat{e}_3, t_1], \quad q_\mu \delta_2^\mu = [\hat{e}_2, \delta_2], \quad q_\mu d_3^\mu = [\hat{e}_3, d_3]. \quad (37)$$

Using these, the divergence of the three-nucleon bound state current is given by

$$\begin{aligned} q_\mu j^\mu &= \bar{\Phi}_1^K P_{12} \delta_1 \delta_2 d_3 ([\hat{e}_2, t_1] + [\hat{e}_3, t_1]) \delta_2 d_3 P_{12} \Phi_1^Q \\ &\quad - \bar{\Phi}_1^K \delta_1 ([\hat{e}_2, \delta_2] d_3 + \delta_2 [\hat{e}_3, d_3]) P_{12} \Phi_1^Q. \end{aligned} \quad (38)$$

Using the bound state equations, Eqs. (3) and (22), and the fact that  $[\hat{e}_3, \delta_2] = [\hat{e}_3, P_{12}] = 0$ , Eq. (38) reduces immediately down to

$$q_\mu j^\mu = -\bar{\Phi}_1^K P_{12} \delta_1 \delta_2 d_3 \hat{e}_2 \Phi_1^Q + \bar{\Phi}_1^K \delta_1 \delta_2 \hat{e}_2 d_3 P_{12} \Phi_1^Q. \quad (39)$$

Since  $[\hat{e}_2, P_{12}] \neq 0$ , it is not immediately obvious that the last two terms cancel. To show that this is indeed the case, we make use of the fact that

$$[\hat{e}_2, P_{12} t_1 P_{12}] = [\hat{e}_2, t_2] = 0. \quad (40)$$

Then using the bound state equation in the last term of Eq. (39) we obtain that

$$\begin{aligned} \bar{\Phi}_1^K \delta_1 \delta_2 \hat{e}_2 d_3 P_{12} \Phi_1^Q &= -\bar{\Phi}_1^K \delta_1 \delta_2 \hat{e}_2 d_3 P_{12} t_1 \delta_2 d_3 P_{12} \Phi_1^Q = -\bar{\Phi}_1^K \delta_1 d_3 \delta_2 \hat{e}_2 P_{12} t_1 P_{12} \delta_1 d_3 \Phi_1^Q \\ &= -\bar{\Phi}_1^K \delta_1 d_3 P_{12} t_1 P_{12} \delta_2 \hat{e}_2 \delta_1 d_3 \Phi_1^Q = -\bar{\Phi}_1^K P_{12} \delta_2 d_3 t_1 P_{12} \delta_2 \hat{e}_2 \delta_1 d_3 \Phi_1^Q \\ &= \bar{\Phi}_1^K P_{12} \delta_2 \hat{e}_2 \delta_1 d_3 \Phi_1^Q = \bar{\Phi}_1^K P_{12} \delta_1 \delta_2 d_3 \hat{e}_2 \Phi_1^Q. \end{aligned} \quad (41)$$

Using this result in Eq. (39) we obtain the current conservation relation

$$q_\mu j^\mu = 0. \quad (42)$$

### E. Normalization condition

The method for obtaining the normalization condition for bound state wave functions in quantum field theory typically involves the taking of residues of Green functions or t-matrices at the bound state pole, and is similar to what is used in quantum mechanics when the potentials are energy dependent. Here we apply the same idea, but to the quantity  $X$ , in order to determine the specific normalisation condition for the three-body bound state vertex function in the spectator approach. Our starting point is the following identity for  $X$ :

$$X (1 + t_1 \delta_2 d_3 P_{12}) X = X. \quad (43)$$

Using the pole behaviour of  $X$  given by Eq. (16), we see that in the vicinity of the three-body bound state pole, Eq. (43) reduces to

$$i \frac{\bar{\Psi}_2^Q (1 + t_1 \delta_2 d_3 P_{12}) \Phi_1^Q}{Q^2 - M^2} = 1, \quad (44)$$

or

$$i \bar{\Psi}_2^Q \frac{\partial (1 + t_1 \delta_2 d_3 P_{12})}{\partial Q^2} \Phi_1^Q \Big|_{Q^2=M^2} = 1. \quad (45)$$

With the understanding that  $Q^2 = M^2$ , and using Eq. (20), this may also be written as

$$-i \bar{\Phi}_1^Q P_{12} \delta_1 \delta_2 d_3 \frac{\partial (t_1 \delta_2 d_3 P_{12})}{\partial Q^2} \Phi_1^Q = 1, \quad (46)$$

or

$$-i \bar{\Phi}_1^Q P_{12} \delta_1 \delta_2 d_3 \frac{\partial t_1}{\partial Q^2} \delta_2 d_3 P_{12} \Phi_1^Q + i \bar{\Phi}_1^Q \delta_1 \frac{\partial (\delta_2 d_3)}{\partial Q^2} P_{12} \Phi_1^Q = 1. \quad (47)$$

This form of the normalisation condition is especially convenient as it is expressed in terms of the two-body t-matrix  $t_1$  rather than the potential  $v_1$  which results, for example, when the full Green function is used in an identity, similar to Eq. (43), but involving two-body potentials [10].

It is sometimes convenient to express the normalization condition as a four-vector relation by using the replacement

$$\frac{\partial}{\partial Q^\mu} \rightarrow 2Q_\mu \frac{\partial}{\partial Q^2}$$

in the above equations. That this replacement is valid can be easily justified by appealing to Lorentz invariance. In this way Eq. (47), written out in full numerical form, becomes

$$\begin{aligned} & \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) \frac{\partial t(Q - k_1, k_2, p_2)}{\partial Q_\mu} \delta(p_2) d(p_3) P_{12} \Phi_1^Q(k_1, p_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) \frac{\partial d(k_3)}{\partial Q_\mu} P_{12} \Phi_1^Q(k_1, k_2) = 2iQ^\mu \end{aligned} \quad (48)$$

where  $k_3 = Q - k_1 - k_2$  and  $p_3 = Q - k_1 - p_2$ .

## F. Charge conservation

In its usual meaning, charge conservation is a consequence of current conservation. As we have proved current conservation above, charge is naturally conserved in our model. On the other hand, that the conserved charge is equal to the total charge of the physical system does not follow automatically from current conservation, and therefore needs to be checked separately. In particular, what needs to be checked is that

$$j^\mu(Q, Q) = 2eQ^\mu \quad (49)$$

where  $e$  is the physical charge of the three-body bound state. We follow current terminology and also refer to Eq. (49) as a statement of “charge conservation” (in the sense that if  $e$  is indeed the physical charge, then no charge has been “lost” in the model). For an exact solution of field theory, Eq. (49) follows from the fact that  $|Q\rangle$  in Eq. (24) is an eigenstate of the charge operator with eigenvalue  $e$ . In this subsection we show that Eq. (49) also holds in our model where the gauging of equations method has been used for the spectator approach.

The bound state current was given in its explicit form in Eq. (32). We can rewrite this expression for zero momentum transfer, and using only independent momentum variables, as

$$\begin{aligned} j^\mu(Q, Q) = & \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4p_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) t^\mu(Q - k_1, Q - k_1; k_2, p_2) \\ & \delta(p_2) d(p_3) P_{12} \Phi_1^Q(k_1, p_2) \\ & - \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta^\mu(k_2, k_2) d(k_3) P_{12} \Phi_1^Q(k_1, k_2) \\ & - \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) d^\mu(k_3, k_3) P_{12} \Phi_1^Q(k_1, k_2) \end{aligned} \quad (50)$$

where  $k_3 = Q - k_1 - k_2$ ,  $p_3 = Q - k_1 - p_2$ , and where  $t^\mu$  is expressed in terms of the total momenta in the (23) system,  $Q - k_1$  in both the initial and final states, and the momenta of particle 2,  $p_2$  and  $k_2$  for initial and final states, respectively.

Both the gauged Feynman propagator  $d^\mu$  and the gauged on-mass-shell particle propagator  $\delta^\mu$  satisfy the Ward identity [12]:

$$i d^\mu(k_3, k_3) = e_3 \frac{\partial d(k_3)}{\partial k_{3\mu}}, \quad (51)$$

$$i \delta^\mu(k_2, k_2) = e_2 \frac{\partial \delta(k_2)}{\partial k_{2\mu}}. \quad (52)$$

The interaction current  $v^\mu$  is an input to our model and therefore satisfies the two-particle Ward identity by construction. In turn, it can easily be shown that the  $t^\mu$ , as given by Eq. (27), must also satisfy the two-particle Ward identity. For the momentum variables of Eq. (50), this identity reads

$$\begin{aligned} i t^\mu(Q - k_1, Q - k_1; k_2, p_2) = & e_2 \frac{\partial t(Q - k_1, k_2, p_2)}{\partial k_{2\mu}} + \frac{\partial t(Q - k_1, k_2, p_2)}{\partial p_{2\mu}} e_2 \\ & + (e_3 + e_2) \frac{\partial t(Q - k_1, k_2, p_2)}{\partial Q_\mu}. \end{aligned} \quad (53)$$

Substituting Eq. (53) into Eq. (50), we may then use the bound state equations for  $\Phi_1^Q$  and  $\bar{\Phi}_1^Q$  to simplify the terms containing  $\partial t/\partial k_{2\mu}$  and  $\partial t/\partial p_{2\mu}$ . Writing Eq. (51) as

$$id^\mu(k_3, k_3) = (e_3 + e_2) \frac{\partial d(k_3)}{\partial k_{3\mu}} - e_2 \frac{\partial d(k_3)}{\partial k_{3\mu}} = (e_3 + e_2) \frac{\partial d(k_3)}{\partial Q_\mu} + e_2 \frac{\partial d(k_3)}{\partial k_{2\mu}} \quad (54)$$

we may then use it together with Eq. (52) in Eq. (50) to obtain

$$\begin{aligned} ij^\mu(Q, Q) = & \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) (e_3 + e_2) \frac{\partial t(Q - k_1, k_2, p_2)}{\partial Q_\mu} \\ & \delta(p_2) d(p_3) P_{12} \Phi_1^Q(k_1, p_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) (e_3 + e_2) \frac{\partial d(k_3)}{\partial Q_\mu} P_{12} \Phi_1^Q(k_1, k_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_2 \frac{\partial \Phi_1^Q(k_1, k_2)}{\partial k_{2\mu}} \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{\partial \bar{\Phi}_1^Q(k_1, k_2)}{\partial k_{2\mu}} e_2 \delta(k_1) \delta(k_2) d(k_3) P_{12} \Phi_1^Q(k_1, k_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) e_2 \frac{\partial \delta(k_2)}{\partial k_{2\mu}} d(k_3) P_{12} \Phi_1^Q(k_1, k_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) e_2 \frac{\partial d(k_3)}{\partial k_{2\mu}} P_{12} \Phi_1^Q(k_1, k_2). \end{aligned} \quad (55)$$

Using integration by parts, we can write the last three terms of this equation as

$$\int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) e_2 d(k_3) \frac{\partial P_{12} \Phi_1^Q(k_1, k_2)}{\partial k_{2\mu}}.$$

Eq. (50) can then be written as

$$\begin{aligned} ij^\mu(Q, Q) = & \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) (e_3 + e_2 + e_1) \frac{\partial t(Q - k_1, k_2, p_2)}{\partial Q_\mu} \\ & \delta(p_2) d(p_3) P_{12} \Phi_1^Q(k_1, p_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) (e_3 + e_2 + e_1) \frac{\partial d(k_3)}{\partial Q_\mu} P_{12} \Phi_1^Q(k_1, k_2) \\ & + \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_1 \frac{\partial t(Q - k_1, k_2, p_2)}{\partial k_{1\mu}} \\ & \delta(p_2) d(p_3) P_{12} \Phi_1^Q(k_1, p_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) e_1 \frac{\partial d(k_3)}{\partial k_{1\mu}} P_{12} \Phi_1^Q(k_1, k_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_2 \frac{\partial \Phi_1^Q(k_1, k_2)}{\partial k_{2\mu}} \\ & + \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_2 \frac{\partial P_{12} \Phi_1^Q(k_1, k_2)}{\partial k_{2\mu}} \end{aligned} \quad (56)$$

where the charge in the first two terms has been increased to the total charge of the system, and where we used the fact that  $\partial t(Q - k_1, k_2, p_2)/\partial Q_\mu = -\partial t(Q - k_1, k_2, p_2)/\partial k_{1\mu}$  and  $\partial d(k_3)/\partial Q_\mu = -\partial d(k_3)/\partial k_{1\mu}$ . Since the bound state vertex function  $\Phi_1^Q$  is an eigenstate of the total charge  $e_1 + e_2 + e_3$  with eigenvalue  $e$ , a comparison with the normalisation condition, Eq. (48), shows that the first two terms of the above equation give the sought after charge conservation relation. Thus all we need to show now is that the last four terms of Eq. (56) cancel each other. To this end we eliminate  $t$  in the third term on the r.h.s. of Eq. (56) by using integration by parts, and then making use of the bound state equations for  $\Phi_1^Q$  and  $\bar{\Phi}_1^Q$ . In this way we get

$$\begin{aligned} & \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_1 \frac{\partial t(Q - k_1, k_2, p_2)}{\partial k_{1\mu}} \delta(p_2) d(p_3) P_{12} \Phi_1^Q(k_1, p_2) \\ &= \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{\partial \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3)}{\partial k_{1\mu}} e_1 \Phi_Q(k_1, k_2) \\ &+ \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, p_2) \delta(k_1) \delta(p_2) e_1 \frac{\partial d(p_3) P_{12} \Phi_Q(k_1, p_2)}{\partial k_{1\mu}} \end{aligned} \quad (57)$$

where  $p_2$  and  $p_3$  in the last equation can now be replaced by  $k_2$  and  $k_3$ , respectively. That the last two terms of Eq. (57) cancel the last three terms of Eq. (56) can then be seen by using the identities

$$\begin{aligned} & \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) e_1 \frac{\partial d(k_3) P_{12} \Phi_1^Q(k_1, k_2)}{\partial k_{1\mu}} \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) e_1 \frac{\partial d(k_3)}{\partial k_{1\mu}} P_{12} \Phi_1^Q(k_1, k_2) \\ & - \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_2 \frac{\partial \Phi_1^Q(k_1, k_2)}{\partial k_{2\mu}} = 0. \end{aligned} \quad (58)$$

and

$$\begin{aligned} & \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{\partial \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3)}{\partial k_{1\mu}} e_1 \Phi_1^Q(k_1, k_2) \\ & + \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \bar{\Phi}_1^Q(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_2 \frac{\partial P_{12} \Phi_1^Q(k_1, k_2)}{\partial k_{2\mu}} \\ & = \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{\partial \bar{\Phi}_1^Q P_{12}(k_1, k_2) \delta(k_1) \delta(k_2) d(k_3) e_1 \Phi_1^Q(k_1, k_2)}{\partial k_{1\mu}} = 0. \end{aligned} \quad (59)$$

Thus we have shown Eq. (49), which proves charge conservation for our gauged three-nucleon spectator model.

### III. GAUGING THE THREE-NUCLEON SCATTERING EQUATIONS

### A. $\gamma^3\text{He} \rightarrow NNN$

Photodisintegration of the three-nucleon bound state into three free nucleons is described by the electromagnetic  $^3\text{He} \rightarrow NNN$  transition current  $j_0^\mu$  consisting of all possible diagrams for this process (“photodisintegration” here means disintegration due to either an on-mass-shell or an off-mass-shell photon, so the case of electrodisintegration is included). By comparison, the gauged vertex function  $\Phi_1^\mu$  consists of all possible diagrams for photodisintegration where nucleons 2 and 3 are last to interact and where no photons are attached to the outgoing nucleons. As we already have an equation for  $\Phi_1^\mu$ , Eq. (13), all that is necessary to obtain  $j_0^\mu$  is to add the missing terms in  $\Phi_1^\mu$ . Indeed, we can immediately write down that

$$j_0^\mu = P_c \left( \Phi_1^\mu + \sum_{i=1}^3 \Gamma_i^\mu d_i \Phi_1 \right) \quad (60)$$

where  $P_c$  is the operator which sums over all the cyclic permutations of the three particle labels. The role of  $P_c$  is to include diagrams where nucleons other than 2 and 3 are last to interact. The term  $P_c \sum_i \Gamma_i^\mu d_i \Phi_1$  consists of all possible diagrams where photons are attached to the final state external legs.

Denoting the three-particle Feynman propagator by  $G_0$ ,

$$G_0 = d_1 d_2 d_3, \quad (61)$$

then

$$G_0^{-1} G_0^\mu = G_0^{-1} (d_1^\mu d_2 d_3 + d_1 d_2^\mu d_3 + d_1 d_2 d_3^\mu) = \sum_{i=1}^3 \Gamma_i^\mu d_i, \quad (62)$$

and Eq. (60) can also be written as

$$j_0^\mu = P_c G_0^{-1} [G_0 \Phi_1]^\mu = P_c \left( G_0^{-1} G_0^\mu \Phi_1 + \Phi_1^\mu \right), \quad (63)$$

indicating that  $j_0^\mu$  can be obtained directly by gauging the quantity  $G_0 \Phi_1$ . This is just what one might expect since  $G_0 \Phi_1$  corresponds to all possible diagrams for the  $^3\text{He} \rightarrow NNN$  process where nucleons (23) are last to interact. In this respect, it is interesting to note that although  $^3\text{He} \rightarrow NNN$  is not a possible physical process, it can nevertheless be gauged to yield a physical electromagnetic process. It is also worth pointing out that although we gauge on-mass-shell propagators when they correspond to internal lines, only Feynman propagators are used in the gauging of the external lines in Eq. (63). This is not inconsistent with the spectator approach, it preserves gauge invariance [11], and it avoids introduction of on-mass-shell  $\delta$ -function like singularities into the physical photodisintegration amplitude.

### B. $Nd \rightarrow \gamma Nd$

We can obtain the amplitude for the process  $Nd \rightarrow \gamma Nd$  by gauging the scattering amplitude for  $Nd \rightarrow Nd$ . Thus our first task is to derive an expression for this amplitude.

From Eq. (13) it is clear that the quantity  $X = (1 + t_1 \delta_2 d_3 P_{12})^{-1}$  describes all possible perturbation graphs for the process  $NNN \rightarrow NNN$  where the 13 pair is the first and the 23 pair is the last to interact (see also Fig. 3). By taking appropriate residues of  $X$  we can therefore obtain any scattering amplitude involving three nucleons, including the one for  $Nd$  elastic scattering. This we now proceed to do.

As seen explicitly in Fig. 3, the second iteration of either of the Eqs. (15) yields a connected graph. We can thus write

$$X = (1 + t_1 \delta_2 d_3 P_{12})^{-1} = 1 - t_1 \delta_2 d_3 P_{12} + X_c \quad (64)$$

where

$$X_c = (1 + t_1 \delta_2 d_3 P_{12})^{-1} t_1 \delta_2 d_3 P_{12} t_1 \delta_2 d_3 P_{12} \quad (65)$$

is the connected part of  $X$ . Using the fact that

$$t_1 \delta_2 d_3 P_{12} t_1 \delta_2 d_3 P_{12} = t_1 d_3 P_{12} t_1 P_{12} \delta_1 \delta_2 d_3 \quad (66)$$

and

$$(1 + t_1 \delta_2 d_3 P_{12})^{-1} t_1 = t_1 (1 + \delta_2 d_3 P_{12} t_1)^{-1}, \quad (67)$$

we may write

$$X = (1 + t_1 \delta_2 d_3 P_{12})^{-1} = 1 - t_1 \delta_2 d_3 P_{12} + T_c P_{12} \delta_1 \delta_2 d_3 \quad (68)$$

where

$$T_c = t_1 (1 + \delta_2 d_3 P_{12} t_1)^{-1} d_3 P_{12} t_1 \quad (69)$$

is the connected part of the scattering amplitude for  $NNN \rightarrow NNN$  where the 23 pair is both first and last to interact. It is easy to see that the corresponding Bethe-Salpeter amplitude is given by

$$T_c^{\text{BS}} = t_1 (1 + d_2 d_3 P_{12} t_1)^{-1} d_3 P_{12} t_1 \quad (70)$$

showing explicitly that the spectator equation expression of Eq. (69) can be obtained from the Bethe-Salpeter expression of Eq. (70) by replacing the spectator particle's propagators by the on-mass-shell propagator in each term of the perturbation series for  $T_c^{\text{BS}}$ .

The two-nucleon t-matrix  $t_1$  contains the deuteron bound state pole. In the vicinity of this pole we have that

$$t_1(P; p_2, k_2) \sim i \frac{\phi_{23}(p_2) \bar{\phi}_{23}(k_2)}{P^2 - M_d^2} = i \frac{\phi_{23} \bar{\phi}_{23}}{P^2 - M_d^2} \quad (71)$$

where  $P$  is the deuteron four-momentum,  $M_d$  is the deuteron mass, and  $\phi_{23}$  is the deuteron vertex function for nucleons 2 and 3. The scattering amplitude  $T_{dd}$  for  $Nd \rightarrow Nd$  is then obtained from Eq. (69) by taking left and right residues at the deuteron pole:



$$T_{dd} = \bar{\phi}_{23}(1 + \delta_2 d_3 P_{12} t_1)^{-1} d_3 P_{12} \phi_{23}. \quad (72)$$

The electromagnetic  $Nd \rightarrow Nd$  transition current  $j_{dd}^\mu$  that describes the process  $Nd \rightarrow \gamma Nd$  can now be obtained as in the four-dimensional case [10] by gauging  $d_1 T_{dd} d_1$ . Defining

$$Y = (1 + \delta_2 d_3 P_{12} t_1)^{-1}, \quad (73)$$

we therefore have that

$$\begin{aligned} j_{dd}^\mu &= d_1^{-1} \left( \bar{\phi}_{23} d_1 Y d_3 P_{12} d_1 \phi_{23} \right)^\mu d_1^{-1} \\ &= \left( \bar{\phi}_{23} Y d_3 P_{12} \phi_{23} \right)^\mu + \bar{\phi}_{23} \Gamma_1^\mu d_1 Y d_3 P_{12} \phi_{23} + \bar{\phi}_{23} Y d_3 d_2 \Gamma_2^\mu P_{12} \phi_{23} \end{aligned} \quad (74)$$

where the first term on the r.h.s. is given by

$$\left( \bar{\phi}_{23} Y d_3 P_{12} \phi_{23} \right)^\mu = \bar{\phi}_{23}^\mu Y d_3 P_{12} \phi_{23} + \bar{\phi}_{23} Y^\mu d_3 P_{12} \phi_{23} + \bar{\phi}_{23} Y d_3^\mu P_{12} \phi_{23} + \bar{\phi}_{23} Y d_3 P_{12} \phi_{23}^\mu. \quad (75)$$

The gauged vertex functions  $\phi_{23}^\mu$  and  $\bar{\phi}_{23}^\mu$  can be obtained by gauging the two-body bound state equations

$$\phi_{23} = \frac{1}{2} v_1 \delta_2 d_3 \phi_{23}, \quad \bar{\phi}_{23} = \frac{1}{2} \bar{\phi}_{23} \delta_2 d_3 v_1. \quad (76)$$

Using the equations for  $t_1$ , Eqs. (26), one easily obtains that

$$\phi_{23}^\mu = \frac{1}{2} \left( 1 + \frac{1}{2} t_1 \delta_2 d_3 \right) (v_1 \delta_2 d_3)^\mu \phi_{23}, \quad \bar{\phi}_{23}^\mu = \frac{1}{2} \bar{\phi}_{23} (\delta_2 d_3 v_1)^\mu \left( 1 + \frac{1}{2} \delta_2 d_3 t_1 \right). \quad (77)$$

To obtain an expression for  $Y^\mu$ , we first note that  $Y$  satisfies the equations

$$Y = 1 - \delta_2 d_3 P_{12} t_1 Y; \quad Y = 1 - Y \delta_2 d_3 P_{12} t_1. \quad (78)$$

Gauging either of these equations then gives

$$\begin{aligned} Y^\mu &= -Y (\delta_2 d_3 P_{12} t_1)^\mu Y \\ &= -Y (\delta_2^\mu d_3 P_{12} t_1 + \delta_2 d_3^\mu P_{12} t_1 + \delta_2 d_3 P_{12} t_1^\mu) Y. \end{aligned} \quad (79)$$

In this way the transition current  $j_{dd}^\mu$  is completely determined in terms of one- and two-body input quantities. Note that our expression for  $j_{dd}^\mu$  is in terms of the quantity  $Y$  rather than the  $X$  introduced earlier. Yet it turns out that once the integrals over the fourth components are taken in the expression for  $j_{dd}^\mu$ , then it is seen that the use of  $X$  or  $Y$  in Eq. (74) is completely equivalent. This is discussed in the Appendix where we show how our four-dimensional expressions of the spectator approach are reduced to three-dimensional forms suitable for numerical calculations.

### C. $\gamma^3\text{He} \rightarrow Nd$

To find the  $^3\text{He} \rightarrow Nd$  electromagnetic transition current  $j_d^\mu$ , it would be natural to simply take the left residue of the  $^3\text{He} \rightarrow NNN$  electromagnetic transition current  $j_0^\mu$ , given in Eq. (60), at the two-body bound state pole of nucleons 2 and 3. Although this is straightforward, one can obtain exactly the same result in an even simpler way by gauging the scattering amplitude  $T_d$  for the off-shell process  $^3\text{He} \rightarrow Nd$ . The expression for  $T_d$  is easily found from the bound state equation for  $\Phi_1$  in Eq. (3) by taking the left residue at the two-body bound state pole:

$$T_d = -\bar{\phi}_{23}\delta_2d_3P_{12}\Phi_1. \quad (80)$$

To make sure that one includes the case where photons are attached to the free nucleon in the final  $Nd$  state (particle 1), it is sufficient to gauge  $d_1T_d$  and then multiply from the left by the inverse propagator  $d_1^{-1}$  at the end. Thus the electromagnetic transition current  $j_d^\mu$  which describes the physical process  $\gamma^3\text{He} \rightarrow Nd$  is given by

$$\begin{aligned} j_d^\mu &= -d_1^{-1}(d_1\bar{\phi}_{23}\delta_2d_3P_{12}\Phi_1)^\mu \\ &= -\left(\Gamma_1^\mu d_1\bar{\phi}_{23}\delta_2d_3 + \bar{\phi}_{23}^\mu\delta_2d_3 + \bar{\phi}_{23}\delta_2^\mu d_3 + \bar{\phi}_{23}\delta_2d_3^\mu\right)P_{12}\Phi_1 - \bar{\phi}_{23}\delta_2d_3P_{12}\Phi_1^\mu, \end{aligned} \quad (81)$$

where all quantities have been specified above.

## IV. SUMMARY

We have derived relativistic three-dimensional integral equations describing the interaction of the three-nucleon system with an external electromagnetic field. In particular, we have presented expressions for the three-nucleon bound state electromagnetic current, as well as for the transition currents describing the scattering processes  $\gamma^3\text{He} \rightarrow NNN$ ,  $Nd \rightarrow \gamma Nd$ , and  $\gamma^3\text{He} \rightarrow Nd$ . Our equations are gauge invariant and conserve charge. More importantly, gauge invariance and charge conservation are achieved in the theoretically correct fashion; namely, by the attachment of photons to all possible places within the strong interaction model of the three nucleons.

The achievement of these results was made possible by the recent development of the gauging of equations method [10]. Previously this method was used to generate a four-dimensional gauge invariant description of the three-nucleon system and its electromagnetic currents. Here we applied the same method to what in principle is an even more challenging problem, namely, the gauging of the spectator equations for the three-nucleon system [4]. The extra difficulty in this case comes from the question of how to choose the spectator particles once the gauging of the four-dimensional equations is done. We solved this problem by (i) working in terms of Faddeev components, and (ii) by introducing the idea of an on-mass-shell nucleon propagator  $\delta$  in order to express the three-nucleon spectator equations in a four-dimensional form [11]. Once in this form, the spectator equations were then gauged directly, in this way allowing the gauging method itself to determine the spectator particles in the final gauged equations.

An important ingredient in our gauged equations is the gauged on-mass-shell propagator  $\delta^\mu$ . The question of how to construct a form for  $\delta^\mu$  that satisfies both the Ward-Takahashi identity and the Ward identity was previously answered in Ref. [11] and [12]. As both  $\delta$  and

$\delta^\mu$  contain on-mass-shell  $\delta$ -functions, our gauged four-dimensional equations can be reduced to a three-dimensional form. The details of this reduction were presented in the Appendix. As a result, we have brought together all the theoretical results that are necessary for a practical calculation of the electromagnetic processes of the three-nucleon system.

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## APPENDIX:

In the main part of this paper, all our results have been expressed in terms of four-dimensional integrals despite the presence of  $\delta$ -functions which could allow us to reduce the integrals to three-dimensional ones. This has been done specifically so that we can follow the gauging procedure introduced in Ref. [10] for four-dimensional integral equations. Our final results, however, are three-dimensional, and it is the purpose of this appendix to write out some of the obtained expressions in a purely three-dimensional form.

We begin with the bound state equation of Eq. (3). This symbolic equation represents the four-dimensional integral equation

$$\Phi_1^Q(p_1, p_2) = - \int \frac{d^4 k_2}{(2\pi)^4} t(Q - p_1; p_2, k_2) \delta(k_2) d(Q - p_1 - k_2) P_{12} \Phi_1^Q(p_1, k_2). \quad (\text{A1})$$

Because of the presence of the on-mass-shell propagator  $\delta(k_2)$ , the integral over  $dk_2^0$  may be done trivially. Setting the momenta  $p_1$  and  $p_2$  to be on mass shell we then obtain the three-dimensional equation

$$\Phi_1^Q(\bar{p}_1, \bar{p}_2) = - \int \frac{d^3 k_2}{(2\pi)^3} t(Q - \bar{p}_1; \bar{p}_2, \bar{k}_2) \frac{\Lambda(\bar{k}_2)}{2\omega_{k_2}} d(Q - \bar{p}_1 - \bar{k}_2) P_{12} \Phi_1^Q(\bar{p}_1, \bar{k}_2) \quad (\text{A2})$$

where  $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$  and  $\bar{k} = (\omega_k, \mathbf{k})$ . Although this equation is three-dimensional, the quantities involved still retain their Dirac spinor structure. Thus, for example,  $\Phi_1^Q(\bar{p}_1, \bar{p}_2)$  consists of a direct product of three Dirac spinors, one for each nucleon, while  $t(Q - \bar{p}_1; \bar{p}_2, \bar{k}_2)$  is a  $16 \times 16$  matrix. For the on-mass-shell nucleons we may eliminate the Dirac spinor structure by appropriate multiplication by the Dirac spinors  $u$  or  $\bar{u}$ . We therefore define

$$\tilde{\Phi}_1^Q(\mathbf{p}_1 \alpha_1, \mathbf{p}_2 \alpha_2) = \bar{u}(\mathbf{p}_1, \alpha_1) \bar{u}(\mathbf{p}_2, \alpha_2) \Phi_1^Q(\bar{p}_1, \bar{p}_2) \quad (\text{A3})$$

and

$$\tilde{t}(Q - \bar{p}_1; \mathbf{p}_2 \alpha_2; \mathbf{k}_2 \beta_2) = \bar{u}(\mathbf{p}_2, \alpha_2) t(Q - \bar{p}_1; \bar{p}_2, \bar{k}_2) u(\mathbf{k}_2, \beta_2). \quad (\text{A4})$$

Since

$$\Lambda(\bar{p}) = \not{p} + m = 2m \sum_{\alpha} u(\mathbf{p}, \alpha) \bar{u}(\mathbf{p}, \alpha), \quad (\text{A5})$$

where the normalization of the Dirac spinors is given by  $\bar{u}(\mathbf{p}, \alpha)u(\mathbf{p}, \beta) = \delta_{\alpha\beta}$ , Eq. (A2) can be transformed to the equation

$$\tilde{\Phi}_1^Q(\mathbf{p}_1\alpha_1, \mathbf{p}_2\alpha_2) = - \sum_{\beta_2} \int \frac{d^3k_2}{(2\pi)^3} \frac{m}{\omega_{k_2}} \tilde{t}(Q - \bar{p}_1; \mathbf{p}_2\alpha_2, \mathbf{k}_2\beta_2) \tilde{d}(Q, \mathbf{p}_1, \mathbf{k}_2) P_{12} \tilde{\Phi}_1^Q(\mathbf{p}_1\alpha_1, \mathbf{k}_2\beta_2) \quad (\text{A6})$$

where

$$\tilde{d}(Q, \mathbf{p}_1, \mathbf{k}_2) = d(Q - \bar{p}_1 - \bar{k}_2), \quad (\text{A7})$$

with the integral being taken over the Lorentz invariant phase space volume  $\frac{d^3k_2}{(2\pi)^3} \frac{m}{\omega_{k_2}}$ . We shall write this three-dimensional equation in symbolic form as

$$\tilde{\Phi}_1^Q = -\tilde{t}_1 \tilde{d}_3 P_{12} \tilde{\Phi}_1^Q. \quad (\text{A8})$$

Note that Eq. (A8) can be considered as an operator expression in three-dimensional momentum space and is written in terms of tilde quantities to distinguish it from Eq. (3) which is a symbolic equation in four-dimensional momentum space.

Similarly, the first scattering equation of Eqs. (15) is a symbolic expression for a four-dimensional integral equation which, after the trivial integration over  $dk_2^0$ , gives

$$X(Q; p_1 p_2; q_1 q_2) = (2\pi)^8 \delta^4(p_1 - q_1) \delta^4(p_2 - q_2) - \int \frac{d^3k_2}{(2\pi)^3} t(Q - p_1; p_2, \bar{k}_2) \frac{\Lambda(\bar{k}_2)}{2\omega_{k_2}} d(Q - p_1 - \bar{k}_2) P_{12} X(Q; p_1 \bar{k}_2, q_1 q_2). \quad (\text{A9})$$

Putting  $p_1$  and  $p_2$  on mass shell in this equation, it can be noticed that the inhomogeneous term becomes

$$(2\pi)^8 \delta^4(\bar{p}_1 - q_1) \delta^4(\bar{p}_2 - q_2) = (2\pi)^8 \delta^3(\mathbf{p}_1 - \mathbf{q}_1) \delta^3(\mathbf{p}_2 - \mathbf{q}_2) \delta(\bar{q}_1^0 - q_1^0) \delta(\bar{q}_2^0 - q_2^0). \quad (\text{A10})$$

This in turn implies that

$$X(Q; \bar{p}_1 \bar{p}_2; q_1 q_2) = X'(Q; \bar{p}_1 \bar{p}_2; \bar{q}_1 \bar{q}_2) \frac{m}{\omega_{q_1}} \frac{m}{\omega_{q_2}} (2\pi)^2 \delta(\bar{q}_1^0 - q_1^0) \delta(\bar{q}_2^0 - q_2^0) \quad (\text{A11})$$

where  $X'(Q; \bar{p}_1 \bar{p}_2; \bar{q}_1 \bar{q}_2)$  satisfies the three-dimensional equation

$$X'(Q; \bar{p}_1 \bar{p}_2; \bar{q}_1 \bar{q}_2) = (2\pi)^6 \frac{\omega_{q_1}}{m} \frac{\omega_{q_2}}{m} \delta^3(\mathbf{p}_1 - \mathbf{q}_1) \delta^3(\mathbf{p}_2 - \mathbf{q}_2) - \int \frac{d^3k_2}{(2\pi)^3} t(Q - \bar{p}_1; \bar{p}_2, \bar{k}_2) \frac{\Lambda(\bar{k}_2)}{2\omega_{k_2}} d(Q - \bar{p}_1 - \bar{k}_2) P_{12} X'(Q; \bar{p}_1 \bar{k}_2, \bar{q}_1 \bar{q}_2). \quad (\text{A12})$$

Note that Eq. (A11) provides the key result that has enabled us to eliminate the zero-component  $\delta$ -functions from the inhomogeneous term.

We now eliminate the Dirac spinor structure of on-mass-shell particles by defining

$$\tilde{X}(Q; \mathbf{p}_1\alpha_1, \mathbf{p}_2\alpha_2; \mathbf{q}_1\beta_1, \mathbf{q}_2\beta_2) = \bar{u}(\mathbf{p}_1, \alpha_1) \bar{u}(\mathbf{p}_2, \alpha_2) X'(Q; \bar{p}_1 \bar{p}_2, \bar{q}_1 \bar{q}_2) u(\mathbf{q}_1, \beta_1) u(\mathbf{q}_2, \beta_2) \quad (\text{A13})$$

In this way Eq. (A12) gets transformed into a three-dimensional equation, similar to Eq. (A6):

$$\begin{aligned} \tilde{X}(Q; \mathbf{p}_1\alpha_1, \mathbf{p}_2\alpha_2; \mathbf{q}_1\beta_1, \mathbf{q}_2\beta_2) &= \delta_{\alpha_1\beta_1} \delta_{\alpha_2\beta_2} (2\pi)^6 \frac{\omega_{q_1}}{m} \frac{\omega_{q_2}}{m} \delta^3(\mathbf{p}_1 - \mathbf{q}_1) \delta^3(\mathbf{p}_2 - \mathbf{q}_2) \\ &- \sum_{\gamma_2} \int \frac{d^3k_2}{(2\pi)^3} \frac{m}{\omega_{k_2}} \tilde{t}(Q - \bar{p}_1; \mathbf{p}_2\alpha_2, \mathbf{k}_2\gamma_2) \tilde{d}(Q, \mathbf{p}_1, \mathbf{k}_2) P_{12} \tilde{X}(Q; \mathbf{p}_1\alpha_1, \mathbf{k}_2\gamma_2; \mathbf{q}_1\beta_1, \mathbf{q}_2\beta_2). \end{aligned} \quad (\text{A14})$$

Since the momentum phase space integration volume includes a factor  $(2\pi)^{-3}m/\omega$ , the inhomogeneous term in this equation acts like a unit operator in momentum space. Thus we can write this equation in symbolic form as

$$\tilde{X} = 1 - \tilde{t}_1 \tilde{d}_3 P_{12} \tilde{X}. \quad (\text{A15})$$

It then follows that  $\tilde{X} = (1 + \tilde{t}_1 \tilde{d}_3 P_{12})^{-1}$  and therefore

$$\tilde{X} = 1 - \tilde{X} \tilde{t}_1 \tilde{d}_3 P_{12}. \quad (\text{A16})$$

Alternatively, one can show Eq. (A16) by starting from the numerical form of the four-dimensional equation  $X = 1 - X t_1 \delta_2 d_3$ , setting the final momenta  $p_1$  and  $p_2$  to be on mass shell, and using Eq. (A11) as before. This time, however, one also needs to use the relation

$$\bar{u}(\mathbf{p}_1, \alpha_1) \bar{u}(\mathbf{p}_2, \alpha_2) X'(Q; \bar{p}_1 \bar{p}_2, \bar{q}_1 \bar{q}_2) = \sum_{\beta_1 \beta_2} \tilde{X}(Q; \mathbf{p}_1\alpha_1, \mathbf{p}_2\alpha_2; \mathbf{q}_1\beta_1, \mathbf{q}_2\beta_2) \bar{u}(\mathbf{q}_1, \beta_1) \bar{u}(\mathbf{q}_2, \beta_2) \quad (\text{A17})$$

which can be proved by using Eqs. (A12) and (A14) to show that each side of Eq. (A17) satisfies the same integral equation.

Note that the bound state and scattering equations of Eq. (A8) and Eq. (A15) have the same kernel and can therefore be calculated using similar numerical codes. It would therefore be convenient to express the amplitude for  $Nd$  scattering,  $T_{dd}$ , in terms of  $\tilde{X}$  rather than leaving it in terms of  $Y$  in which it is given in Eq. (72). To this end we first write  $Y$  in terms of  $X$  by using the definitions of Eq. (14) and Eq. (73):

$$Y = 1 - \delta_2 d_3 P_{12} X t_1. \quad (\text{A18})$$

Then the  $Nd$  amplitude is given by

$$T_{dd} = \bar{\phi}_{23} Y d_3 P_{12} \phi_{23} \quad (\text{A19})$$

$$= \bar{\phi}_{23} (1 - \delta_2 d_3 P_{12} X t_1) d_3 P_{12} \phi_{23}. \quad (\text{A20})$$

As particle 1 in the final state is on mass shell, we see that both particles 1 and 2 to the left of  $X$  in the latter equation are on mass shell. We can therefore use Eq. (A11) and Eq. (A17) in Eq. (A20) to write the physical  $Nd$  scattering amplitude  $\tilde{T}_{dd} = \bar{u}_1 T_{dd} u_1$  in the three-dimensional form

$$\tilde{T}_{dd} = \bar{\phi}_{23} (1 - \tilde{d}_3 P_{12} \tilde{X} \tilde{t}_1) \tilde{d}_3 P_{12} \phi_{23} \quad (\text{A21})$$

where  $\tilde{\phi}_{23} = \bar{u}_2 \phi_{23}$ , and  $\bar{\phi}_{23} = \bar{\phi}_{23} u_2$ . Writing this equation as

$$\tilde{T}_{dd} = \tilde{\phi}_{23} \tilde{d}_3 P_{12} (1 - \tilde{X} \tilde{t}_1 \tilde{d}_3 P_{12}) \tilde{\phi}_{23}, \quad (\text{A22})$$

we may then use Eq. (A16) to obtain that

$$\tilde{T}_{dd} = \tilde{\phi}_{23} \tilde{d}_3 P_{12} \tilde{X} \tilde{\phi}_{23}. \quad (\text{A23})$$

We now show how the physical transition current  $\tilde{j}_{dd}^\mu = \bar{u}_1 j_{dd}^\mu u_1$  can be similarly expressed in terms of three-dimensional expressions involving  $\tilde{X}$ . In terms of  $Y$ , the expression for  $j_{dd}^\mu$  was given in Eq. (74) which we can write in the form

$$\begin{aligned} j_{dd}^\mu = & \bar{\phi}_{23} Y d_3 P_{12} (\Gamma_3^\mu d_3 \phi_{23} + \phi_{23}^\mu) + \bar{\phi}_{23}^\mu Y d_3 P_{12} \phi_{23} - \bar{\phi}_{23} Y d_3 P_{12} (\Gamma_3^\mu d_3 t_1 + t_1^\mu) \delta_1 Y d_3 P_{12} \phi_{23} \\ & - \bar{\phi}_{23} Y d_3 P_{12} t_1 \delta_1^\mu Y d_3 P_{12} \phi_{23} + \bar{\phi}_{23} \Gamma_1^\mu d_1 Y d_3 P_{12} \phi_{23} + \bar{\phi}_{23} Y d_3 P_{12} d_1 \Gamma_1^\mu \phi_{23}. \end{aligned} \quad (\text{A24})$$

Every  $Y$  in this equation comes in the combination  $Y d_3 P_{12}$ , just as in Eq. (A19) for  $T_{dd}$ . In each of the first three terms on the r.h.s. of Eq. (A24) (upper line), every  $Y$  has the first particle on its left and the second particle on its right restricted to be on mass shell. Thus we may proceed as for Eq. (A19) and replace our expressions by three-dimensional ones where  $Y d_3 P_{12}$  is replaced by  $\tilde{d}_3 P_{12} \tilde{X}$ . For example the first term becomes

$$\bar{u}_1 \bar{\phi}_{23} Y d_3 P_{12} (\Gamma_3^\mu d_3 \phi_{23} + \phi_{23}^\mu) u_1 = \tilde{\phi}_{23} \tilde{d}_3 P_{12} \tilde{X} (\Gamma_3^\mu \tilde{d}_3 \tilde{\phi}_{23} + \tilde{\phi}_{23}^\mu). \quad (\text{A25})$$

On the other hand, in the last three terms of Eq. (A24) every  $Y$  has either an off mass shell first particle on its left or an off mass shell second particle on its right. For example, in the last term of Eq. (A24) particle 2 to the right of  $Y$  is off mass shell. Proceeding nevertheless as before, we have that

$$\begin{aligned} \bar{u}_1 \bar{\phi}_{23} Y d_3 P_{12} d_1 \Gamma_1^\mu \phi_{23} u_1 &= \bar{u}_1 \bar{\phi}_{23} (1 - \delta_2 d_3 P_{12} X t_1) d_3 P_{12} d_1 \Gamma_1^\mu \phi_{23} u_1 \\ &= \bar{\phi}_{23} d_3 P_{12} d_1 \Gamma_1^\mu u_1 \tilde{\phi}_{23} - \tilde{\phi}_{23} d_3 P_{12} \tilde{X} t_1^L d_3 P_{12} d_1 \Gamma_1^\mu \tilde{\phi}_{23} \end{aligned} \quad (\text{A26})$$

where again we have used Eqs. (A11) and (A17) to reduce the integrals to three-dimensions. In the last equation,  $t_1$  has only its left particle 2 on mass shell, hence the notation  $t_1^L$ . Thus, although Eq. (A26) is three-dimensional and in terms of  $\tilde{X}$ , it cannot be simplified further with the help of Eq. (A16). What can be done, however, is to use the second of the equations for  $t_1$ , Eqs. (26), in order to express  $t_1^L$  in terms of  $\tilde{t}_1$  and  $v_1^L$ ; indeed, it's easily seen that  $t_1^L$  is given by the three-dimensional equation

$$t_1^L = v_1^L + \frac{1}{2} \tilde{t}_1 \tilde{d}_3 v_1^L. \quad (\text{A27})$$

With the help of Eq. (A16), this enables one to re-express Eq. (A26) in terms of  $v_1^L$ .

A similar three-dimensional reduction holds for the second last term in Eq. (A24). The first term in the last line of Eq. (A24) contains the gauged on-mass-shell propagator  $\delta_1^\mu$ . As seen from Eq. (9),  $\delta_1^\mu$  has one of its legs on mass shell while the other is off mass shell. Thus one of the terms  $Y d_3 P_{12}$  in the fourth term of Eq. (A24) can be simplified by using Eq. (A16), while the other one cannot.

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